PHYS 301 - Assignment #2 Sol'ns - 20241002

1. Require $\vec{\nabla}\times\vec{E} = 0$ in electrostatics

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

(i) $\vec{E} = k\left[xy\,\hat{x} + 2yz\,\hat{y} + 3xz\,\hat{z} \right]$

$$\frac{\vec{\nabla}\times\vec{E}}{k} = \hat{x}\left(\frac{\partial}{\partial y}(3xz) - \frac{\partial}{\partial z}(2yz)\right)$$

$$-\hat{y}\left(\frac{\partial}{\partial x}(3xz) - \frac{\partial}{\partial z}(xy)\right)$$

$$+\hat{z}\left(\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial y}(xy)\right)$$

$$= -2y\,\hat{x} - 3z\,\hat{y} - x\,\hat{z} \neq 0$$

$\therefore$ (i) is not a valid electrostatic $\vec{E}$-field.

(ii) $$\frac{\vec{\nabla}\times\vec{E}}{k} = \hat{x}\left(\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(2xy+z^2)\right)$$
$$-\hat{y}\left(\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(y^2)\right)$$
$$+\hat{z}\left(\frac{\partial}{\partial x}(2xy+z^2) - \frac{\partial}{\partial y}(y^2)\right)$$

$$= \hat{x}(2z-2z) - \hat{y}(0-0) + \hat{z}(2y-2y)$$
$$= 0 \checkmark$$ valid electrostatic $\vec{E}$-field.

(b) Know $-\vec{\nabla}V = \vec{E}$

$$\therefore -\frac{\partial V}{\partial x} = E_x = ky^2$$

unknown fcn of y, z.

$$\therefore V = -kxy^2 + f(y,z)$$

$$\therefore E_y = -\frac{\partial V}{\partial y} = +2\cancel{k}xy - \frac{\partial f}{\partial y} = 2\cancel{kx}y + kz^2$$

$$\therefore \quad \frac{\partial f}{\partial y} = -kz^2 \Rightarrow f = -kyz^2 + g(z)$$

unknown fcn of z.

$$\therefore V = -kxy^2 - kyz^2 + g(z)$$

$$E_z = -\frac{dV}{dz} = \cancel{2kyz} + \frac{dg}{dz} = \cancel{2kyz}$$

$$\therefore \frac{dg}{dz} = 0 \Rightarrow g = c' \text{ (const).}$$

$$\boxed{\therefore V = -k\left(xy^2 + yz^2 + C\right)}$$

$$\left[C = -\frac{c'}{k}\right]$$

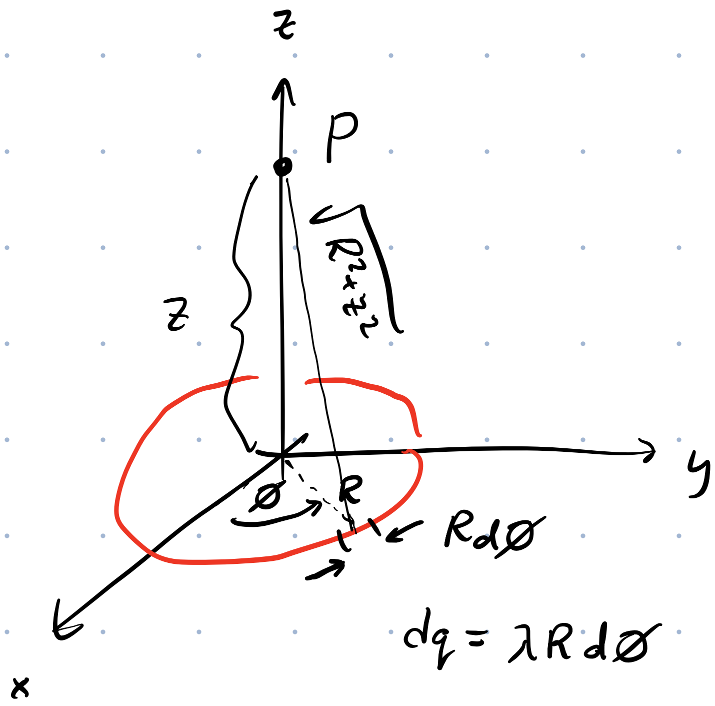
just another const.

check:

$$\vec{E} = -\vec{\nabla} V = k\left[y^2\,\hat{x} + \left(2xy + z^2\right)\hat{y} + 2yz\,\hat{z}\right] \checkmark$$

2. (a) For a uniformly-charged ring of radius R,

$$\lambda = \frac{Q}{2\pi R}$$



dV due to dq is:

$$dV = \frac{1}{4\pi\varepsilon_0} \frac{dq}{\sqrt{R^2+z^2}}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{\lambda R\, d\phi}{\sqrt{R^2+z^2}}$$

$$\therefore V(z) = \int dV = \int_{\phi=0}^{2\pi} \frac{1}{4\pi\varepsilon_0} \frac{\lambda R}{\sqrt{R^2+z^2}}\, d\phi$$

$$\boxed{\therefore V(z) = \frac{1}{4\pi\varepsilon_0} \frac{(\lambda\, 2\pi R)}{\sqrt{R^2+z^2}} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{\sqrt{R^2+z^2}}}$$

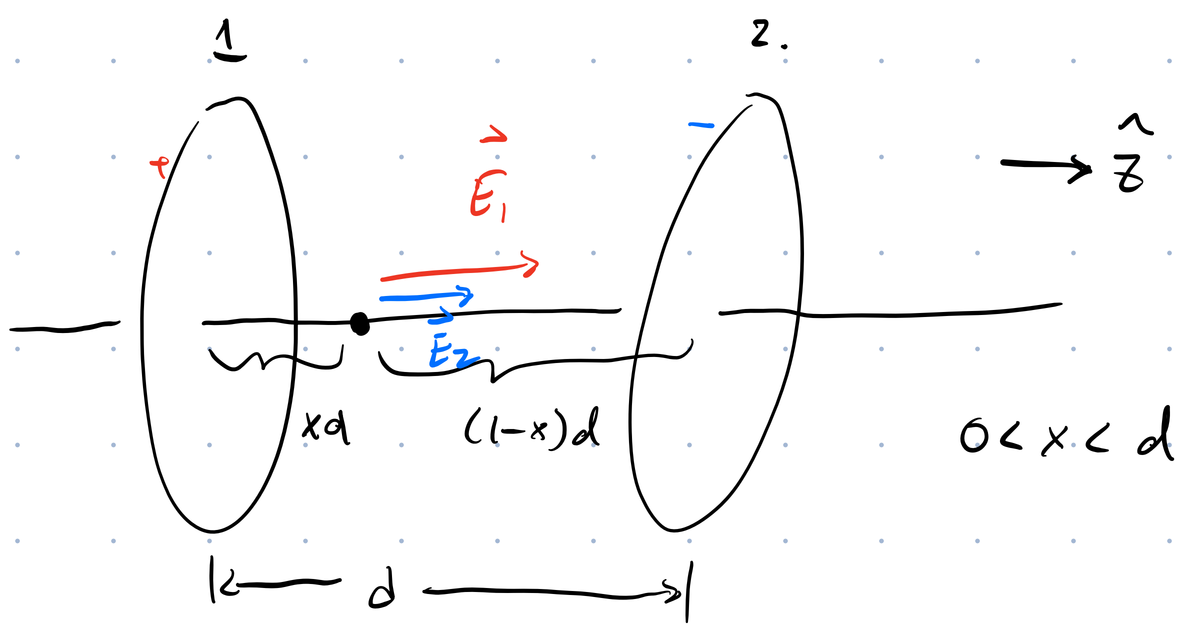
$(\lambda\, 2\pi R) \leftarrow Q$

(b) $\vec{E} = -\frac{dV}{dz}\hat{z}$

$$\frac{dV}{dz} = \frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{2}\right)\frac{2z}{(R^2+z^2)^{3/2}}$$

$$\boxed{\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Qz}{(R^2+z^2)^{3/2}}\hat{z}}$$

(c)



For ring 1 $\quad z = xd \quad \therefore E_1 = \frac{1}{4\pi\varepsilon_0}\frac{Qxd}{(R^2+(xd)^2)^{3/2}}$

$$\therefore \vec{E}_1 = \frac{1}{4\pi\varepsilon_0 R^3} \frac{Q x d}{\left[1+\left(x\frac{d}{R}\right)^2\right]^{3/2}} \hat{z}$$

$$= \underbrace{\frac{Q}{4\pi\varepsilon_0 R^2}}_{\equiv E_0} \frac{x\frac{d}{R}\ \hat{z}}{\left[1+\left(x\frac{d}{R}\right)^2\right]^{3/2}} = E_0 \frac{x\, d/R\ \hat{z}}{\left[1+\left(x\, d/R\right)^2\right]^{3/2}}$$

For ring 2, $z = (1-x)d$

$$\therefore \vec{E}_2 = \frac{1}{4\pi\varepsilon_0} \frac{Q(1-x)d}{\left[R^2+\left((1-x)d\right)^2\right]^{3/2}} \hat{z}$$

$$= E_0 \frac{(1-x)\, d/R}{\left[1+\left((1-x)\, d/R\right)^2\right]^{3/2}} \hat{z}$$

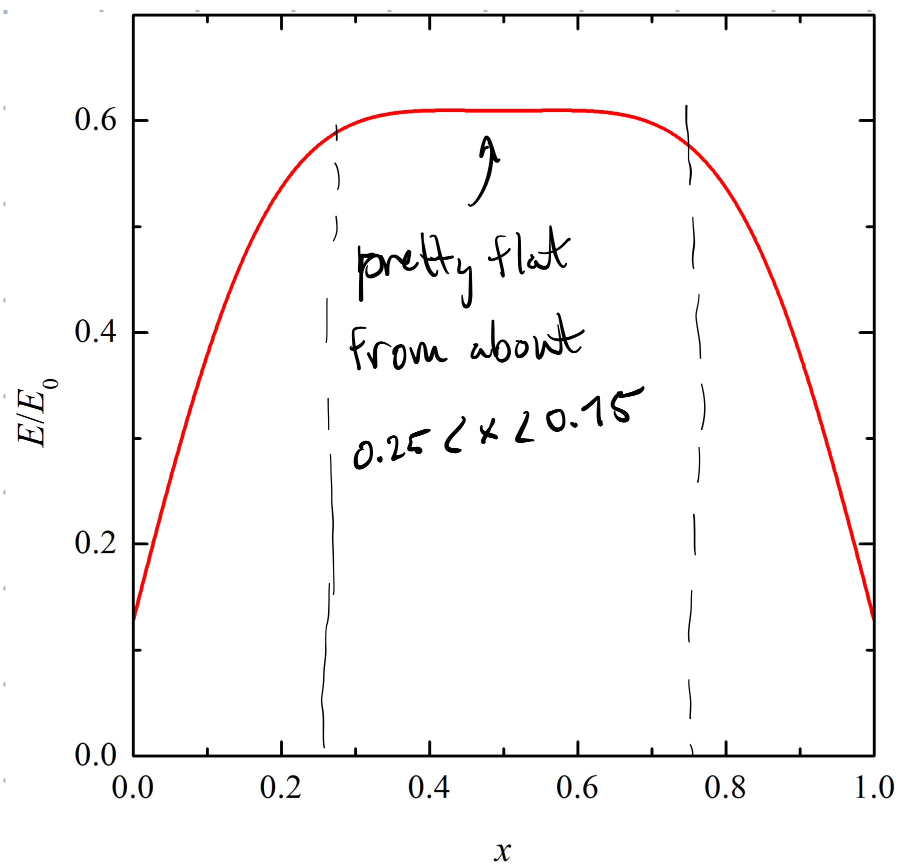
$\vec{E}_1$ & $\vec{E}_2$ both point along $+\hat{z}$-dir'n.

$\therefore$ their contributions simple add.

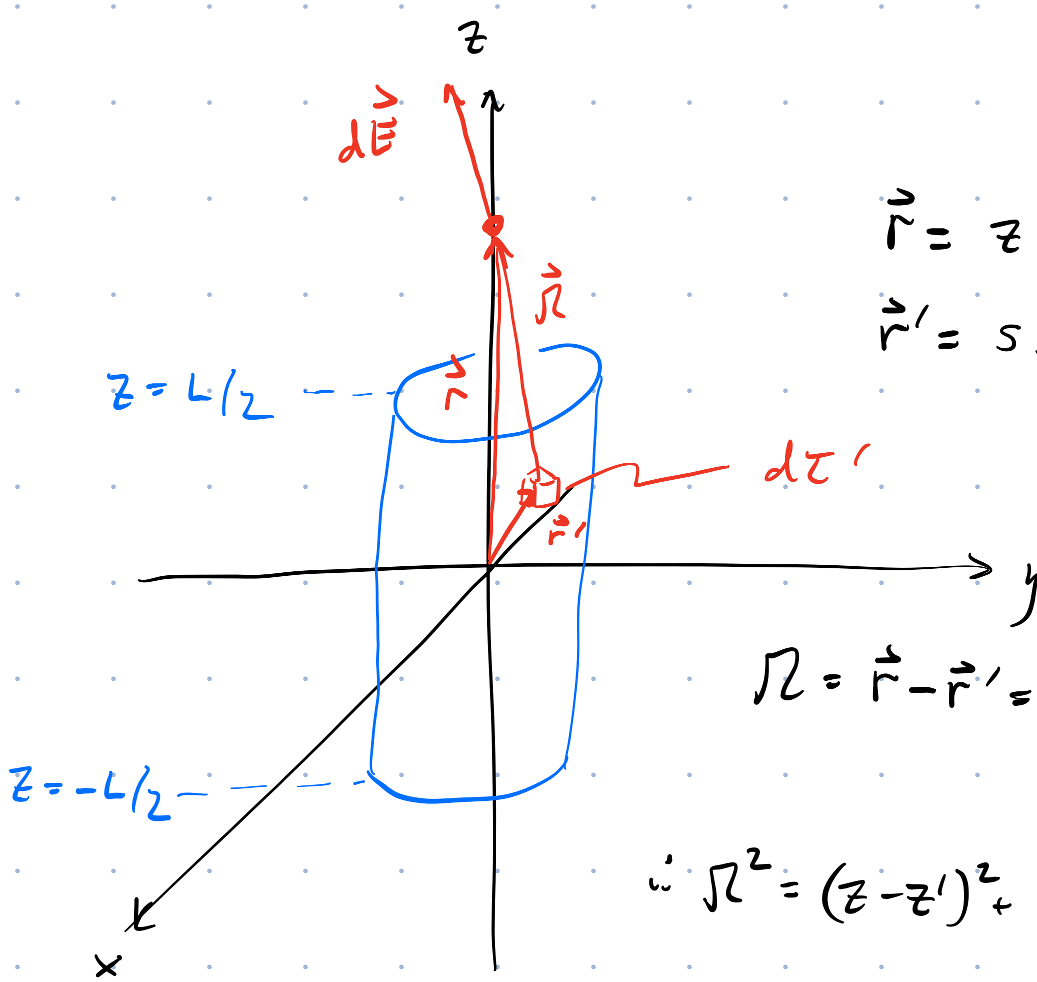
$$\boxed{\begin{aligned}\vec{E} &= \vec{E}_1 + \vec{E}_2 \\ &= E_0 \left\{ \frac{x\,\frac{d}{R}}{\left[1+\left(x\,d/R\right)^2\right]^{3/2}} + \frac{(1-x)\,\frac{d}{R}}{\left[1+\left((1-x)\frac{d}{R}\right)^2\right]^{3/2}} \right\} \hat{z}\end{aligned}}$$

(d) For $d = 2.5R$

$$\frac{E}{E_0} = \frac{2.5x}{\left[1+2.5^2x^2\right]^{3/2}} + \frac{2.5(1-x)}{\left[1+2.5^2(1-x)^2\right]^{3/2}}$$



3 (a)



$\vec{r} = z\,\hat{z}$

$\vec{r}' = s\,\hat{s} + z'\,\hat{z}$

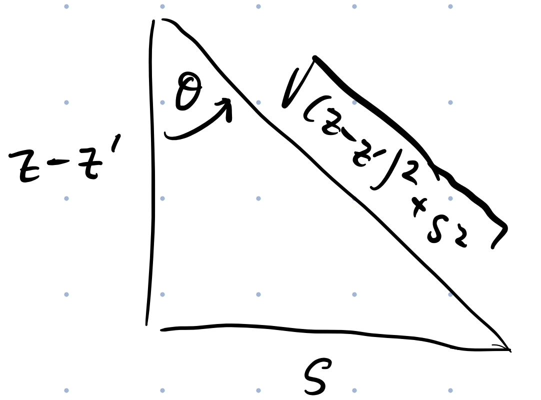
$\mathscr{r} = \vec{r} - \vec{r}' = (z - z')\,\hat{z} - s\,\hat{s}$

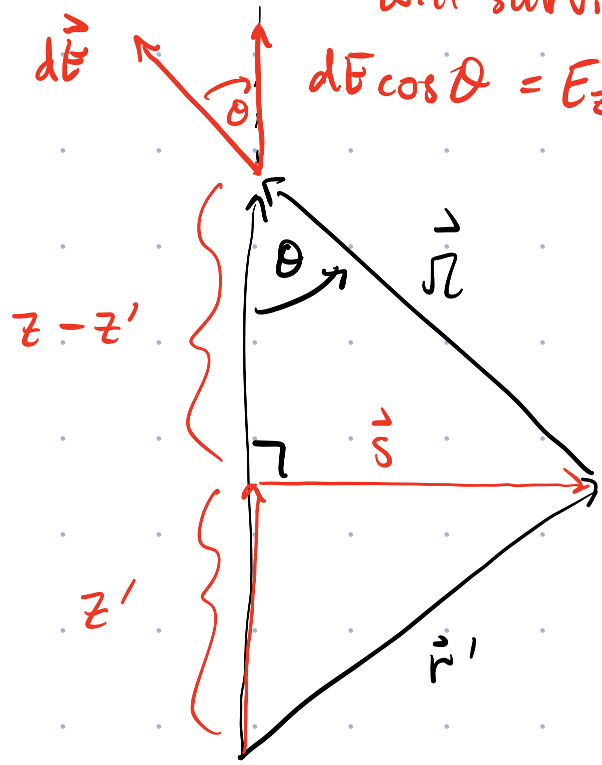
$\therefore \mathscr{r}^2 = (z - z')^2 + s^2$

Side View

only z-component will survive.

$dE\cos\theta = E_z$





$$\cos\theta = \frac{z - z'}{\sqrt{(z-z')^2 + s^2}}$$

Note $\rho(\vec{r}') = \rho$ (const).

$$\therefore E_z = \frac{\rho}{4\pi\varepsilon_0} \int \frac{\cos\theta}{\mathscr{r}^2} \, d\tau'$$

$$d\tau' = s' \, ds' \, dz \, d\phi$$

$$\therefore E_z = \frac{\rho}{4\pi\varepsilon_0} \underbrace{\int_{\phi=0}^{2\pi} d\phi}_{2\pi} \int_{z=-L/2}^{L/2} \int_{s=0}^{R} \frac{(z-z') \, s' \, ds' \, dz}{\left[(z-z')^2 + s^2\right]^{3/2}}$$

Evaluate s integral.

Sub. $u = (z-z')^2 + s^2$

$du = 2s\,ds \Rightarrow s\,ds = \frac{du}{2}$

when $s=0$, $u = (z-z')^2$

$s=R$, $u = (z-z')^2 + R^2$

$$E_z = \frac{\rho}{2\varepsilon_0}\int_{z=-L/2}^{L/2}\int_{u=(z-z')^2}^{(z-z')^2+R^2} (z-z')\, u^{-3/2}\, \frac{du}{2}\, dz'$$

$$= \frac{\rho}{4\varepsilon_0}\int_{z=-L/2}^{L/2} (z-z')(-2)\, u^{-1/2}\Big|_{(z-z')^2}^{(z-z')^2+R^2} dz'$$

$$= \frac{-\rho}{2\varepsilon_0}\int_{z=-L/2}^{L/2} (z-z')\left[\frac{1}{\sqrt{(z-z')^2+R^2}} - \frac{1}{z-z'}\right] dz'$$

$$= -\frac{\rho}{2\varepsilon_0}\left[\int_{-L/2}^{L/2} \frac{(z-z')}{\sqrt{(z-z')^2+R^2}}\, dz' - \underbrace{\int_{-L/2}^{L/2} dz'}_{L}\right]$$

another sub.

$$w = (z-z')^2 + R^2$$

$$dw = -2(z-z')\,dz' \qquad \therefore\ (z-z')\,dz' = -\frac{dw}{2}$$

$z' = -L/2 \qquad w = \left(z+\frac{L}{2}\right)^2 + R^2$

$z' = L/2 \qquad w = \left(z-\frac{L}{2}\right)^2 + R^2$

$$\therefore E_z = \frac{\rho}{4\varepsilon_0} \int_{w=(z+\frac{L}{2})^2+R^2}^{(z-\frac{L}{2})^2+R^2} w^{-1/2}\, dw + \frac{\rho L}{2\varepsilon_0}$$

$$= \frac{\rho}{2\varepsilon_0} w^{1/2} \Big|_{(z+\frac{L}{2})^2+R^2}^{(z-\frac{L}{2})^2+R^2} + \frac{\rho L}{2\varepsilon_0}$$

$$\boxed{\therefore \vec{E} = \frac{\rho}{2\varepsilon_0}\left[\sqrt{\left(z-\frac{L}{2}\right)^2+R^2} - \sqrt{\left(z+\frac{L}{2}\right)^2+R^2} + L\right]\hat{z}}$$

(b) Far away from the cylinder $(z \gg L, R)$, expect $\vec{E}$ to look like a pt. charge field.

$$Q = \rho \pi R^2 L \qquad \therefore \rho = \frac{Q}{\pi R^2 L}$$

$$E = \frac{Q}{2\pi\varepsilon_0 R^2 L}\left[\sqrt{\left(z-\frac{L}{2}\right)^2+R^2} - \sqrt{\left(z+\frac{L}{2}\right)^2+R^2} + L\right]$$

Want to show that everything in [ ] reduces to $\frac{R^2 L}{2z^2}$ when $z \gg R, L$.

Strategy is to use repeated application of the binomial approx.

$$(1+x)^n \approx 1 + nx$$

for $|x| \ll 1$.

consider

$$\sqrt{\left(z \pm \frac{L}{2}\right)^2 + R^2}$$

$$= \left(z \pm \frac{L}{2}\right) \underbrace{\sqrt{1 + \left(\frac{R}{z \pm \frac{L}{2}}\right)^2}}_{\approx 1 + \frac{1}{2}\left(\frac{R}{z \pm \frac{L}{2}}\right)^2}$$

since z large.

$$= \left(z \pm \frac{L}{2}\right)\left[1 + \frac{1}{2}\left(\frac{R}{z \pm \frac{L}{2}}\right)^2\right]$$

$$= \left(z \pm \frac{L}{2}\right)\left[1 + \frac{L}{2z^2}\left(\frac{R}{1 \pm \frac{L}{2z}}\right)^2\right]$$

$$= \left(z \pm \frac{L}{2}\right)\left[1 + \frac{R^2}{2z^2}\underbrace{\left(1 \pm \frac{L}{2z}\right)^{-2}}_{\approx 1 \mp \frac{L}{z} \text{ since } z \text{ large.}}\right]$$

$$= \left(z \pm \frac{L}{2}\right)\left[1 + \frac{R^2}{2z^2}\left(1 \mp \frac{L}{z}\right)\right]$$

$$= \left(z \pm \frac{L}{2}\right)\left(1 + \frac{R^2}{2z^2} \mp \frac{R^2 L}{2z^3}\right)$$

$$= z + \frac{R^2}{2z} \mp \frac{R^2 L}{2z^2} \pm \frac{L}{2} \pm \frac{R^2 L}{4z^2} - \frac{R^2 L^2}{4z^3}$$

$$\therefore \sqrt{\left(z - \frac{L}{2}\right)^2 + R^2} - \sqrt{\left(z + \frac{L}{2}\right)^2 + R^2}$$

$$= \cancel{z} + \cancel{\frac{R^2}{2z}} + \frac{R^2 L}{2z^2} - \frac{L}{2} - \frac{R^2 L}{4z^2} - \cancel{\frac{R^2 L^2}{4z^3}}$$

$$- \left(\cancel{z} + \cancel{\frac{R^2}{2z}} - \frac{R^2 L}{2z^2} + \frac{L}{2} + \frac{R^2 L}{4z^2} - \cancel{\frac{R^2 L^2}{4z^3}}\right)$$

$$= -L + \frac{R^2 L}{z^2} - \frac{R^2 L}{2z^2}$$

$$= -L + \frac{R^2 L}{2z^2}$$

Finally...

$$E_z \approx \frac{Q}{2\pi\varepsilon_0 \cancel{R^2 L}} \left[\cancel{-L} + \frac{\cancel{R^2 L}^{\,1}}{2z^2} + \cancel{L} \right]$$

or $$\boxed{E_z \approx \frac{Q}{4\pi\varepsilon_0 z^2}}$$

as expected $\vec{E}$ is like a pt. charge when $z \gg L, R$.